Stability of a Time-varying Fishing Model with Delay

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Abstract We introduce a delay differential equation model which describes how fish are harvested

$$\dot{N}(t) = \left[\frac{a(t)}{1 + \left(\frac{N(\theta(t))}{K(t)} \right)^{\gamma}} - b(t) \right] N(t) \qquad (A)$$

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In our previous studies we investigated the persistence of equation (A) and existence of a periodic solution for this equation.

Here we study the stability (local and global) of the periodic solutions of equation (A).

Keywords-Fishery, Periodic Environment, Delay Differential Equations, Global and Local Stability.

1 Introduction and Preliminaries

Consider the following differential equation which is widely used in Fisheries [1, 2]

$$\dot{N} = [\beta(t,N) - M(t,N)]N, \tag{1}$$

where $N = N(t)$ is the population biomass, $\beta(t,N)$ is the per-capita fecundity rate, and $M(t,N)$ is the per-capita fishing mortality rate due to natural mortality causes and harvesting.

In equation (1) let $\beta(t,N)$ be a Hill's type function [1, 2, 5]

$$\beta(t,N) = \frac{a}{1+\left(\frac{N}{K}\right)^{\gamma}}, \tag{2}$$

where a and K are positive constants, and $\gamma > 0$ is a parameter.

Traditional Population Ecology is based on the concept that carrying capacity does not change over time even though it is known [3] that the values of carrying capacity related to the habitat areas might vary, e.g., year-to-year changes in weather affect fish population.

We assume that in (2) $a = a(t)$, $K = K(t)$, and $M(t,N) = b(t)$ are continuous positive functions.

Generally, Fishery models [1, 2] recognize that for real organisms it takes time to develop from newborns to reproductively active adults.

Let in equation (2) $N = N(\theta(t))$, where $\theta(t)$ is the maturation time delay $0 \le \theta(t) \le t$. If we take into account that delay, then we have the following time-lag model based on equation (1)

$$\dot{N}(t) = \left[\frac{a(t)}{1+\left(\frac{N(\theta(t))}{K(t)}\right)^{\gamma}} - b(t)\right] N(t) \tag{3}$$

for $\gamma > 0$, with the initial function and the initial value

$$N(t) = \varphi(t),\ t < 0,\ N(0) = N_0 \tag{4}$$

under the following conditions:

(a1) $a(t), b(t), K(t)$ are continuous on $[0, \infty)$ functions, $b(t) \geq b > 0,\ K \geq K(t) \geq k > 0$;

(a2) $\theta(t)$ is a continuous function, $\theta(t) \leq t,\ \limsup_{t\to\infty} \theta(t) = \infty$;

(a3) $\varphi : (-\infty, 0) \to R$ is a continuous bounded function, $\varphi(t) \geq 0, N_0 > 0$.

Definition 1.1 A function $N : R \to R$ with continuous derivative is called *a (global) solution* of problem (3), (4), if it satisfies equation (3) for all $t \in [0, \infty)$ and equalities (4) for $t \leq 0$.

If t_0 is the first point, where the solution $N(t)$ of (3), (4) vanishes, i.e., $N(t_0) = 0$, then we consider the only positive solutions of the problem (3), (4) on the interval $[0, t_0)$.

Recently [5] we proved the following results:

Lemma 1.1 *Suppose $a(t) > b(t)$,*

$$\sup_{t>0} \int_{\theta(t)}^{t} (a(s) - b(s))ds < \infty,\ \ \sup_{t>0} \int_{\theta(t)}^{t} b(s)ds < \infty.$$

Then there exists the global positive solution of (3), (4) and this solution is persistence:

$$0 < \alpha_N \leq N(t) \leq \beta_N < \infty.$$

Lemma 1.2 *Let $a(t), b(t), K(t), \theta(t)$ be T-periodic functions, $a(t) \geq b(t)$. If at least one of the following conditions hold:*

(b1)

$$\inf_{t\geq 0} \left(\frac{a(t)}{b(t)} - 1 \right) K^{\gamma}(t) > 1,$$

(b2)

$$\sup_{t\geq 0} \left(\frac{a(t)}{b(t)} - 1 \right) K^{\gamma}(t) < 1,$$

then equation (3) has at least one periodic positive solution $N_0(t)$.

In what follows, we use a classical result from the theory of differential equations with delay [4, 6].

Lemma 1.3 *Suppose that for linear delay differential equation*

$$\dot{x}(t) + r(t)x(h(t)) = 0 \tag{5}$$

where $0 \le t - h(t) \le \sigma$, *the following conditions hold:*

$$r(t) \ge r_0 > 0, \tag{6}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} r(s)ds < \frac{3}{2} \tag{7}$$

Then for every solution x *of equation(5) we have* $\lim_{t\to\infty} x(t) = 0$.

2 Main Results

Let us study global stability of the periodic solutions of equation (3).

Theorem 2.1 *Let* $a(t), b(t), K(t), \theta(t)$ *be T-periodic functions, satisfying conditions of Lemma 1.1 and one of conditions b1) or b2) of Lemma 1.2. Suppose also that*

$$a(t) \ge a_0 > 0, \quad \gamma \int\limits_{\theta(t)}^{t} a(s)ds < 6. \tag{8}$$

Then there exists the unique positive periodic solution $N_0(t)$ *of (3) and for every positive solution N(t) of (3) we have*

$$lim_{t\to\infty}(N(t) - N_0(t)) = 0,$$

i.e., the positive periodic solution $N_0(t)$ *is a global attractor for all positive solutions of (3).*

Proof. Lemma 1.2 implies that there exists a positive periodic solution $N_0(t)$. If that solution is an attractor for all positive solutions then it is the unique positive periodic solution.

We set $N(t) = \exp(x(t))$ and rewrite equation (3) in the form

$$\dot{x}(t) = \frac{a(t)}{1 + \left(\frac{e^{x(\theta(t))}}{K(t)}\right)^{\gamma}} - b(t). \tag{9}$$

Suppose $u(t)$ and $v(t)$ are two different solutions of (9). Denote $w(t) = u(t) - v(t)$. To prove the Theorem 2.1 it is sufficient to show that $\lim_{t\to\infty} w(t) = 0$.

It follows

$$\dot{w}(t) = a(t)\left[\frac{1}{1+\left(\frac{e^{u(\theta(t))}}{K(t)}\right)^{\gamma}} - \frac{1}{1+\left(\frac{e^{v(\theta(t))}}{K(t)}\right)^{\gamma}}\right) \tag{10}$$

Let

$$f(y,t) = \frac{1}{1+\left(\frac{e^{y}}{K(t)}\right)^{\gamma}}. \tag{11}$$

Using the mean value theorem, we have for every t

$$f(y,t) - f(z,t) = f'(c)(y-z), \tag{12}$$

where $\min\{y,z\} \le c(t) \le \max\{y,z\}$.

Clearly,

$$f'_y(y,t) = -\frac{\gamma\left(\frac{e^{y}}{K(t)}\right)^{\gamma}}{\{1+\left(\frac{e^{y}}{K(t)}\right)^{\gamma}\}^2} \tag{13}$$

and $\left|f'_y(y,t)\right| < \frac{1}{4}\gamma$.

Equalities (11)- (12) imply that equation (10) takes the form

$$\dot{w}(t) = -M(t)w(\theta(t)), \tag{14}$$

where

$$M(t) = \frac{\gamma a(t)\left(\frac{e^{c(t)}}{K(t)}\right)^{\gamma}}{\{1+\left(\frac{e^{c(t)}}{K(t)}\right)^{\gamma}\}^2},$$

and

$$\min\{u(\theta(t)), v(\theta(t))\} \le c(t) \le \max\{u(\theta(t)), v(\theta(t))\}.$$

Now we want to check that for equation(14) all conditions of Lemma 1.3 hold.

From (13) we have $M(t) < \frac{1}{4}\gamma a(t)$. Therefore inequality (7) holds. Let us check inequality (6). Set $N_1(t) = e^{u(t)},\ N_2(t) = e^{v(t)}$, where $N_1(t), N_2(t)$ are two solutions of equation (3), corresponding to the solutions $u(t)$ and $v(t)$ of equation (9). Lemma 1.1 implies that

$$M(t) \ge \frac{\gamma a_0\left(\frac{\min\{\alpha_{N_1},\alpha_{N_2}\}}{K}\right)^{\gamma}}{\left(1+\left(\frac{\max\{\beta_{N_1},\beta_{N_2}\}}{k}\right)^{\gamma}\right)^2} > 0,$$

where α_N and β_N are defined by Lemma 1.1. Hence inequality (6) holds and therefore Theorem 2.1 is proven.

Consider now equation (3) with proportional coefficients:

$$\dot{N}(t) = \left[\frac{ar(t)}{1 + \left(\frac{N(\theta(t))}{K}\right)^{\gamma}} - br(t) \right) N(t), \tag{15}$$

where $r(t) \geq r_0 > 0$. Clearly, if $a > b$ then equation (15) has the unique positive equilibrium

$$N^* = \left(\frac{a}{b} - 1\right)^{\frac{1}{\gamma}} K. \tag{16}$$

Corollary 1. If $a > b, \left(\frac{a}{b} - 1\right) K^{\gamma} \neq 1, r(t) \geq r_0 > 0$, and

$$\gamma a \limsup_{t \to \infty} \int_{\theta(t)}^{t} r(s)ds < 6, \tag{17}$$

then the equilibrium N^* is a global attractor for all positive solutions of equation (15).

Let us now compare the global attractivity condition (17) with the local stability conditions.

Theorem 2.2 *Suppose $a > b, r(t) \geq r_0 > 0$ and*

$$\frac{\gamma(a-b)b}{a} \limsup_{t \to \infty} \int_{\theta(t)}^{t} r(s)ds < \frac{3}{2}. \tag{18}$$

Then the equilibrium N^* of equation(15) is locally asymptotically stable.

Proof. Set $x = N - N^*$ and from equation (15) we have

$$\dot{x}(t) = \left[\frac{ar(t)}{1 + \left(\frac{x(\theta(t))+N^*}{K}\right)^{\gamma}} - br(t) \right] (x(t) + N^*). \tag{19}$$

Denote

$$F(u,v) = \left[\frac{ar(t)}{1 + \left(\frac{u+N^*}{K}\right)^{\gamma}} - br(t) \right] (v + N^*).$$

Clearly,

$$F'_u(0,0) = -\frac{\gamma(a-b)b}{a} r(t)$$

and $F'_v(0,0) = 0$. Hence for equation (15) the linearized equation has a form

$$\dot{x}(t) = -\frac{\gamma(a-b)b}{a} r(t)x(\theta(t)). \tag{20}$$

Lemma 1.3 and condition (18) imply that equation (20) is asymptotically stable, therefore the positive equilibrium N^* of equation (15) is locally asymptotically stable.

Compare now Theorem 2.1 and Theorem 2.2. We have $\max\{b(a-b)\} = a/4$. Therefore, if

$$a\gamma \limsup_{t\to\infty} \int_{\theta(t)}^{t} r(s)ds < 6,$$

then equation (15) has locally asymptotically stable equilibrium N^*.

The last condition does not depend on b, and is identical to condition (17) that guarantees the existence of a global attractor. Therefore in Theorem 2.2 we obtained the best possible conditions for global attractivity for equation (3).

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